On inhomogeneous Diophantine approximation and Hausdorff dimension

by Michel Laurent

Abstract − Let $\Gamma = \mathbf{Z}A + \mathbf{Z}^n \subset \mathbf{R}^n$ be a dense subgroup with rank $n+1$ and let $\hat{\omega}(A)$ denote the exponent of uniform simultaneous rational approximation to the point A. We show that for any real number $v \geq \hat{\omega}(A)$, the Hausdorff dimension of the set $\mathcal{B}_v$ of points in $\mathbf{R}^n$ which are v-approximable with respect to Γ, is equal to $1/v$.

1. Inhomogeneous approximation.

We first introduce the general framework of inhomogeneous approximation, following the traditional setting employed in the book of Cassels [7], and adhering to the notations of [5] for the various exponents of approximation involved.

Let m and n be positive integers and let A be a $n \times m$ matrix with real entries. The transposed matrix of A is denoted by ${}^tA$. We consider both the subgroup

$$\Gamma = A\mathbf{Z}^m + \mathbf{Z}^n \subset \mathbf{R}^n,$$

generated modulo $\mathbf{Z}^n$ by the m columns of A, and its *dual* subgroup

$$\Gamma' = {}^tA\mathbf{Z}^n + \mathbf{Z}^m \subset \mathbf{R}^m,$$

generated modulo $\mathbf{Z}^m$ by the n rows of A. It may be enlightening to view alternatively Γ as a subgroup of classes modulo $\mathbf{Z}^n$, lying in the n-dimensional torus $\mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n$. Kronecker's theorem asserts that Γ is dense in $\mathbf{R}^n$ iff the dual group Γ' has maximal rank $m+n$ over $\mathbf{Z}$. We shall assume from now that $\mathrm{rk}_{\mathbf{Z}}\,\Gamma' = m+n$.

In order to measure how sharp is the approximation to a given point β in $\mathbf{R}^n$ by elements of Γ, we introduce the following exponent $\omega(A,\beta)$. For any point θ in $\mathbf{R}^n$, denote by $|\theta|$ the supremum norm of θ and by $\|\theta\| = \min_{x\in\mathbf{Z}^n} |\theta - x|$ the distance in $\mathbf{T}^n$ between θ mod $\mathbf{Z}^n$ and 0.

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Definition 1. *For any $\beta \in \mathbf{R}^n$, let $\omega(A,\beta)$ be the supremum, possibly infinite, of the real numbers ω for which there exist infinitely many integer points $q \in \mathbf{Z}^m$ such that*

$$\|Aq - \beta\| \leq |q|^{-\omega}.$$

It is plain from the definition that $\omega(A,\beta) \geq 0$. Now, in relation with the linear independence of the rows of A, we introduce for any real matrix M the following *uniform* homogeneous exponent:

Definition 2. *Let M be an $m \times n$ matrix with real entries. We denote by $\hat{\omega}(M)$ the supremum, possibly infinite, of the real numbers ω such that for any sufficiently large positive real number Q, there exists a non-zero integer point $q \in \mathbf{Z}^n$ such that*

$$|q| \leq Q \quad \text{and} \quad \|Mq\| \leq Q^{-\omega}.$$

Dirichlet's box principle shows that $\hat{\omega}(M) \geq n/m$. We are now able to formulate the classical transfer between homogeneous and inhomogeneous approximation in terms of these exponents thanks to the

Theorem 1 [5]**.** *For any n-tuple β of real numbers, the lower bound*

$$\omega(A,\beta) \geq \frac{1}{\hat{\omega}({}^tA)} \tag{1}$$

holds true. Moreover we have equality of both members in (1) for almost all β with respect to the Lebesgue measure on $\mathbf{R}^n$.

We come now to our main topic which is the study for $v \geq 0$ of the family of subsets

$$\mathcal{B}_v = \{\beta \in \mathbf{R}^n; \quad \omega(A,\beta) \geq v\} \subseteq \mathbf{R}^n,$$

and of their Hausdorff dimension $\delta(v)$ as a function of v. It follows immediately from Theorem 1 that $\mathcal{B}_v = \mathbf{R}^n$ when $v \leq 1/\hat{\omega}({}^tA)$, while $\mathcal{B}_v$ is a null set for $v > 1/\hat{\omega}({}^tA)$. Furthermore, we know that these latter sets are rather small thanks to the following crude result, quoted as Proposition 7 in [5]:

Theorem 2. *For any real number $v > 1/\hat{\omega}({}^tA)$, the Hausdorff dimension $\delta(v)$ is strictly less than n.*

In fact, the proof of Proposition 7 of [5] gives the explicit upper bound

$$\delta(v) \leq n - 1 + \frac{1}{1 + (v\,\hat{\omega}({}^tA) - 1)/(1+v)}. \tag{2}$$

On the other hand, an easy application of Hausdorff-Cantelli Lemma (see [1, 3]) provides us with the following bound:

Theorem 3. *For any $v > 0$, we have*

$$\delta(v) \le \min\left(n, \frac{m}{v}\right). \tag{3}$$

We refer to Theorem 5 of [4] for a proof of the inequality (3). Note that (2) is certainly sharper than (3) when v belongs to the interval $[1/\hat{\omega}({}^tA), m/n]$, while the upper bound (3) is expected to be an equality for sufficiently large values of v. When $m = n = 1$, it has been proved independently in [2] and in [11] that $\delta(v) = \min(1, 1/v)$, so that (3) is indeed an equality for any $v > 0$ in that case. However, the examples displayed in Theorem 1 of [4] for $(m,n) = (2,1)$ or $(m,n) = (3,1)$, show that the inequality (3) may be strict for any given $v > 1$. Motivated by Theorem 5 below, we address the following

Problem. *Assume that $\hat{\omega}(A)$ is finite. Show that $\delta(v) = m/v$ for any v sufficiently large in term of $\hat{\omega}(A)$.*

Notice that $\hat{\omega}(A) \ge m/n$. It seems plausible that the assumption $v \ge \hat{\omega}(A)$ should always be sufficient in order to ensure that $\delta(v) = m/v$. It holds true when $m = 1$ according to Theorem 5 below. Note also that the lower bound $v \ge \hat{\omega}(A)$ occurs naturally in the construction of a Cantor-type set $\mathcal{K}$ as in Section 4.

2. Simultaneous approximation.

Our knowledge concerning the Hausdorff dimension $\delta(v)$ is more substantial for $m = 1$, that is to say when

$$\Gamma = \mathbf{Z}\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{pmatrix} + \mathbf{Z}^n$$

is generated by a single vector spinning in $\mathbf{T}^n$, thanks to the fine results [4] obtained by Bugeaud and Chevallier. With regard to the above Problem, let us first quote their Theorem 3 as follows:

Theorem 4. *Let $A = {}^t(\alpha_1, \dots, \alpha_n)$ be an $n \times 1$ real matrix with $1, \alpha_1, \dots, \alpha_n$ linearly independent over $\mathbf{Q}$. Then $\delta(v) = 1/v$ for any $v \ge 1$.*

We state now our main result.

Theorem 5. *Let $A = {}^t(\alpha_1, \dots, \alpha_n)$ be an $n \times 1$ real matrix with $1, \alpha_1, \dots, \alpha_n$ linearly independent over $\mathbf{Q}$. Then the equality $\delta(v) = 1/v$ holds true for any $v \ge \hat{\omega}(A)$.*

Note that Theorem 5 extends the previous statement since

$$\frac{1}{n} \le \hat{\omega}(A) \le 1.$$

The lower bound $\hat{\omega}(A) \geq 1/n$ follows immediately from Dirichlet's box principle, while the upper bound $\hat{\omega}(A) \leq 1$ is implicitely contained in the seminal work [10] of Khintchine. It is expected that any intermediate value should be reached for some $n \times 1$ matrix A. We direct to [5, 6] for more precise informations on that topic.

Theorem 5 implies the following

Corollary. *Assume that $\hat{\omega}(A) = 1/n$. Then*

$$\delta(v) = \min\left(n, \frac{1}{v}\right)$$

for any $v > 0$.

The above statement was initially established by Bugeaud and Chevallier in [4], under the stronger assumption that A is a *regular* matrix (according to the terminology of [7]), meaning that there exists a positive real number ϵ such that the lower bound

$$\min_{\substack{q \in \mathbf{Z} \\ 0 < q \leq Q}} \|qA\| \geq \epsilon Q^{-1/n}$$

holds for arbitrary large values of Q.

The proof of Theorem 5 is based on the mass distribution principle [3, 9]. This method enables us to bound from below the Hausdorff measure $\mathcal{H}^f(\mathcal{B}_v)$ of the set $\mathcal{B}_v$ for suitable dimension functions f. It turns out that $\mathcal{H}^f(\mathcal{B}_v) = +\infty$ when $f(r) = r^{1/v}\log(r^{-1})$ and $v > \hat{\omega}(A)$, as it can be easily seen with some minor modifications of the proof given in Section 4. Since $v \mapsto 1/v$ is a decreasing function, a standard argument of Hausdorff measure (see [1] p. 71) then shows that the Hausdorff dimension of the smaller subset

$$\mathcal{B}'_v = \{\beta \in \mathbf{R}^n; \quad \omega(A, \beta) = v\} \subseteq \mathbf{R}^n,$$

coincides with the Hausdorff dimension $\delta(v) = 1/v$ of $\mathcal{B}_v$ if $v > \hat{\omega}(A)$. It follows that for fixed A, the set of values of the exponent $\omega(A, \beta)$ contains the whole interval $]\hat{\omega}(A), +\infty[$, when β ranges over $\mathbf{R}^n$.

2. Best approximations.

We review here some properties of the *best approximations* to A which are needed for proving Theorem 5. Their detailled proof can be found in Section 5 of [4] and in [8]. Throughout this section, A stands for a $n \times 1$ matrix.

A best approximation to A is a positive integer q such that $\|pA\| > \|qA\|$ for every integer p with $0 < p < q$. Let $(q_k)_{k \geq 0}$ be the ordered sequence of these best approximations, starting with $q_0 = 1$. Put

$$\rho_k = \min_{0 < q < q_k} \|qA\| = \|q_{k-1}A\|.$$

It is readily observed that $\hat{\omega}(A)$ is equal to the lower limit of the ratio $\log(\rho_k^{-1})/\log q_k$, as $k \to +\infty$. Therefore, if v is any given real number greater than $\hat{\omega}(A)$, the inequality

$$\|q_{k-1}A\| \geq 4q_k^{-v} \tag{4}$$

holds for infinitely many k.

The key point is to remark that, for large k, the set

$$\Gamma_k = \{qA + \mathbf{Z}^n; \quad 0 \leq q < q_k\},$$

when viewed as a subset of $\mathbf{T}^n$, is closed to a finite group Λ_k which is well distributed in the torus. Let P_k be the closest integer point to $q_k A$. Set now

$$\Lambda_k = \{q\frac{P_k}{q_k} + \mathbf{Z}^n; \quad 0 \leq q < q_k\} = \{q\frac{P_k}{q_k} + \mathbf{Z}^n; \quad q \in \mathbf{Z}\}.$$

Clearly Λ_k is lattice in $\mathbf{R}^n$ with determinant q_k^{-1}. Let $\lambda_{1,k} \leq \cdots \leq \lambda_{n,k}$ be the successive minima of the lattice Λ_k with respect to the unit ball $|x| \leq 1$.

Lemma 1. *For any integer k and any ball $B(x,r) \subset \mathbf{R}^n$ centered at the point x with radius r, we have the following upper bounds (†). If $r \leq \lambda_{i,k}$ for some $i \leq n$, then*

$$\mathrm{Card}\Big(\Gamma_k \cap B(x,r)\Big) \ll \prod_{j=1}^{i-1} \frac{r}{\lambda_{j,k}} \ll \Big(q_k \prod_{j=i}^{n} \lambda_{j,k}\Big) r^{i-1},$$

(with the convention that the empty product is equal to 1 when $i=1$). If $r \geq \lambda_{n,k}$, then

$$\mathrm{Card}\Big(\Gamma_k \cap B(x,r)\Big) \ll q_k r^n.$$

Furthermore $\rho_k \asymp \lambda_{1,k}$, and the last minimum $\lambda_{n,k}$ tends to 0 when k tends to infinity.

Proof. We first prove the above inequalities for $x = 0$ with Γ_k replaced by Λ_k. To that purpose, thanks to LLL algorithm, we use a *reduced* basis $\{e_1, \ldots, e_n\}$ of the lattice Λ_k, meaning that $|e_i| \asymp \lambda_{i,k}$ for $1 \leq i \leq n$ and $|\sum x_i e_i| \asymp \max |x_i e_i|$. We easily obtain the expected bounds for Card $\Big(\Lambda_k \cap B(0,r)\Big)$, using morever Minkowski's theorem on successive minima:

$$\prod_{j=1}^{n} \lambda_{j,k} \asymp \det \Lambda_k = q_k^{-1}.$$

(†) The constants involved in the symbols $\ll$ and $\asymp$ depend only on n. The ball $B(x,r)$ denotes the hypercube of points $y \in \mathbf{R}^n$ with $|y - x| \leq r$.

See [4] for more details. Next, the same inequalities hold for any point $x \in \mathbf{R}^n$ since Λ_k is a group. In order to replace finally Λ_k by Γ_k, observe that the distance between the points qA and qP_k/q_k is smaller than $\rho_{k+1} < \rho_k \ll \lambda_{1,k}$, for any integer q with $0 \le q < q_k$.

As for the assertions concerning $\lambda_{1,k}$ and $\lambda_{n,k}$, we refer to §5 of [4]. $\square$

4. Proof of Theorem 5 and of its corollary.

Let us first deduce the corollary from Theorem 5. Thanks to transfer inequalities between uniform exponents due to Apfelbeck and Jarník (see for instance formula (6) in [5]), we know that $\hat{\omega}(A) = 1/n$ iff $\hat{\omega}({}^tA) = n$. Then, it follows from Theorem 1 that $\mathcal{B}_v = \mathbf{R}^n$ when $v \le 1/n$, so that $\delta(v) = n$ for any v in the interval $[0, 1/n]$. On the other hand, Theorem 5 gives $\delta(v) = 1/v$ for $v \ge 1/n$. Therefore, the formula

$$\delta(v) = \min\left(n, \frac{1}{v}\right)$$

holds true for any positive real number v.

As for the proof of Theorem 5, note that the dimension $\delta(v)$ is a non-increasing function of v and that $\delta(v) \le 1/v$ by Theorem 3. Thus, it suffices to establish the lower bound $\delta(v) \ge 1/v$ for any $v > \hat{\omega}(A)$. We closely follow the lines of [4].

Let v and s be positive real numbers such that $v > \hat{\omega}(A)$ and $s < 1/v$. We construct a Cantor-type set $\mathcal{K} \subseteq \mathcal{B}_v$ whose Hausdorff dimension is $\ge s$. Let $(k_j)_{j\ge 0}$ be an increasing sequence of positive integers such that (4) holds for any integer $k = k_j, j \ge 0$, appearing in the sequence. The sequence (k_j) is also assumed to be very lacunary, in the sense that each value k_{j+1} is taken sufficiently large in term of the preceding value k_j. The precise meaning of these growth conditions will be explicited in the course of the construction.

The set $\mathcal{K}$ is the intersection

$$\mathcal{K} = \cap_{j\ge 0} K_j$$

of nested sets K_j. Each K_j is a finite union of closed balls B with radius $q_{k_j}^{-v}$, centered at some point of Γ_{k_j}. Therefore $\mathcal{K}$ is clearly contained in $\mathcal{B}_v$. Note that the K_j are made up with disjoint balls, as a consequence of (4). We start by taking k_0 arbitrary and by choosing for K_0 a single ball of the required type. Put $N_0 = 1$. We define inductively $K_1 \supset K_2 \supset \dots$ as follows. Suppose that K_j has already been constructed. Since the sequence of points $(qA)_{q\ge 1}$ is uniformely distributed modulo $\mathbf{Z}^n$ in $\mathbf{T}^n$ ([7] Chapter IV), we may choose k_{j+1} large enough so that each ball occurring in K_j, whose Euclidean volume is equal to $2^n q_{k_j}^{-nv}$, contains $\sim 2^n q_{k_{j+1}} q_{k_j}^{-nv}$ points of $\Gamma_{k_{j+1}}$. Dropping eventually some of them, we select in each ball B occurring in K_j exactly the same number

$$N_{j+1} = \left[2^{n-1} q_{k_{j+1}} q_{k_j}^{-nv}\right]$$

of points in $B \cap \Gamma_{k_{j+1}}$ for which the balls B' with radius $q_{k_{j+1}}^{-v}$ centered at these points are included in B. We define K_{j+1} as the union of all these selected balls B', for any B in K_j.

We define now a probability measure μ on $\mathbf{R}^n$ in the following way. First, if B is one of the balls which is part of a set K_j, we set

$$\mu(B) = \frac{1}{N_0 \times \cdots \times N_j},$$

so that $\mu(K_j) = 1$. For any borelian subset E, put

$$\mu(E) = \inf_{\mathcal{C}} \Big(\sum_{B \in \mathcal{C}} \mu(B) \Big),$$

where the infimum is taken over all coverings $\mathcal{C}$ of $E \cap \mathcal{K}$ by disjoint balls B occurring in the sets $K_j, j \geq 0$. Then μ is a probability measure on $\mathbf{R}^n$ whose support is contained in $\mathcal{K}$ [9].

Lemma 2. *For any point $x \in \mathbf{R}^n$ and any sufficiently small radius r, we have the upper bound*

$$\mu(B(x,r)) \ll r^s.$$

Proof. Let j be the index determined by

$$q_{k_{j+1}}^{-v} \leq r < q_{k_j}^{-v}.$$

The set $\mathcal{K} \cap B(x,r)$ is certainly covered by the collection of all balls B with radius $q_{k_{j+1}}^{-v}$ involved in K_{j+1} which intersect $B(x,r)$. Therefore

$$(5) \qquad \mu(B(x,r)) \leq \sum_{B \cap B(x,r) \neq \emptyset} \mu(B) \leq \frac{1}{N_0 \times \cdots \times N_{j+1}} \operatorname{Card} \Big(\Gamma_{k_{j+1}} \cap B(x, r + q_{k_{j+1}}^{-v}) \Big).$$

We make use of Lemma 1 to bound the right hand side of (5).

Suppose first that

$$r + q_{k_{j+1}}^{-v} \leq \lambda_{1,k_{j+1}}.$$

Then Lemma 1 (with $i = 1$) gives

$$\mu(B(x,r)) r^{-s} \ll \frac{(q_{k_{j+1}}^{-v})^{-s}}{N_0 \times \cdots \times N_{j+1}} \ll \frac{q_{k_j}^{nv}}{N_0 \times \cdots \times N_j} q_{k_{j+1}}^{sv-1} \ll 1,$$

provided $q_{k_{j+1}} \geq (q_{k_j}^{nv}/(N_0 \times \cdots \times N_j))^{1/(1-sv)}$ (note that the exponent $sv-1$ is negative).

Suppose now that there exists an integer i with $1 \le i < n$, such that

$$\lambda_{i,k_{j+1}} \le r + q_{k_{j+1}}^{-v} \le \lambda_{i+1,k_{j+1}}.$$

We distinguish two cases, depending on whether $i < s$ or $i \ge s$. If $i < s$, using Lemma 1, we get the same bound

$$\mu(B(x,r))r^{-s} \ll \frac{r^{i-s}}{(N_0 \times \cdots \times N_{j+1})(\lambda_{1,k_{j+1}} \times \ldots \times \lambda_{i,k_{j+1}})} \ll \frac{q_{k_j}^{nv}}{N_0 \times \cdots \times N_j} q_{k_{j+1}}^{sv-1},$$

since

$$\lambda_{i,k_{j+1}} \ge \ldots \ge \lambda_{1,k_{j+1}} \gg \rho_{k_{j+1}} \gg q_{k_{j+1}}^{-v} \quad \text{and} \quad r \ge q_{k_{j+1}}^{-v}.$$

When $i \ge s$, Lemma 1 and (5) give the bounds

$$\begin{aligned}
\mu(B(x,r))r^{-s} &\ll \frac{1}{N_0 \times \cdots \times N_{j+1}} r^{i-s} q_{k_{j+1}} \prod_{\ell=i+1}^{n} \lambda_{\ell,k_{j+1}} \\
&\ll \frac{q_{k_j}^{nv}}{(N_0 \times \cdots \times N_j) q_{k_{j+1}}} \lambda_{i+1,k_{j+1}}^{i-s} q_{k_{j+1}} \prod_{\ell=i+1}^{n} \lambda_{\ell,k_{j+1}} \\
&\ll \frac{q_{k_j}^{nv}}{N_0 \times \cdots \times N_j} \lambda_{n,k_{j+1}}^{n-s} \ll 1,
\end{aligned}$$

provided $\lambda_{n,k_{j+1}} \le (q_{k_j}^{nv}/(N_0 \times \cdots \times N_j))^{-1/(n-s)}$. But we also know from Lemma 1 that $\lambda_{n,k_{j+1}}$ is arbitrarily small when k_{j+1} is sufficiently large (note that the exponent $n - s$ is positive since $s < 1/v < 1/\hat{\omega}(A) \le n$).

Suppose finally that

$$r + q_{k_{j+1}}^{-v} \ge \lambda_{n,k_{j+1}}.$$

Recalling that $r \le q_{k_j}^{-v}$, Lemma 1 gives now

$$\begin{aligned}
\mu(B(x,r))r^{-s} &\ll \frac{1}{N_0 \times \cdots \times N_{j+1}} r^{n-s} q_{k_{j+1}} \ll \frac{q_{k_j}^{nv}}{N_0 \times \cdots \times N_j} (q_{k_j}^{-v})^{n-s} \\
&\ll \frac{q_{k_j}^{sv}}{N_0 \times \cdots \times N_j} \ll \frac{q_{k_{j-1}}^{nv}}{N_0 \times \cdots \times N_{j-1}} q_{k_j}^{sv-1} \ll 1,
\end{aligned}$$

provided $q_{k_j} \ge (q_{k_{j-1}}^{nv}/(N_0 \times \cdots \times N_{j-1}))^{1/(1-sv)}$. □

By the mass distribution principle, Lemma 2 ensures that the Hausdorff dimension of $\mathcal{K}$ is greater or equal to s. Since $\mathcal{K} \subseteq \mathcal{B}_v$, it follows that $\delta(v) \ge s$. Taking now s arbitrarily close to $1/v$, we obtain the lower bound $\delta(v) \ge 1/v$. The proof of Theorem 5 is now complete.

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